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SPIN 1/2 PARTICLE IN THE FIELD OF THE DIRAC STRING ON THE BACKGROUND OF DE SITTER SPACE–TIME

The Dirac monopole string is specified for de Sitter cosmological model. Dirac equation for spin 1/2 particle in presence of this monopole has been examined on the background of de Sitter space-time in static coordinates. Instead of spinor monopole harmonics, the technique of Wigner D-functions is used. After separation of the variables, detailed analysis of the radial equations is performed; four types of solutions, singular, regular, in- and out- running waves, are constructed in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\varepsilon,j,m,\lambda}(t,r,\theta,\phi)$ has been constructed, special attention is given to treating the states of minimal values of the total angular moment $j_{\min}$.

Key words: Dirac monopole string, de Sitter space-time, Dirac equation, Wigner D-function, hypergeometric function.

Introduction

De Sitter and anti de Sitter geometrical models are given steady attention in the context of developing quantum theory in a curved space-time – for instance, see in [1, 2]. In particular, the problem of description of the particles with different spins on these curved backgrounds has a long history [3–36]. Here we will be interested mostly in treating the Dirac equation in de Sitter model.

In the present paper, the influence of the Dirac monopole string on the spin 1/2 particle in de Sitter cosmological model is investigated. Such a problem for spinless particle in the flat Minkowski space was first considered by Dirac [37] and Tamm [38]; Harish-Chandra [39] obtained the exact solution of Dirac equation for electron interacting with magnetic-monopole field. Instead of spinor monopole harmonics, the technique of Wigner D-functions is used. After separation of the variables radial equation have been solved exactly in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\varepsilon,j,m,\lambda}(t,r,\theta,\phi)$ has been constructed. Special attention is given to treating the states of minimal values for total angular momentum quantum number $j_{\min}$, these states turn to be much more complicated than in the flat Minkowski space.

1. Dirac particle in de Sitter space

The Dirac equation (the notation according to [40] is used)

$$\left[i\gamma^c (e^{\alpha}_{(c)}\partial_\alpha + \frac{1}{2}\sigma^{ab}\gamma_{abc}) - M \right]\Psi = 0 \qquad (1)$$

in static coordinates and tetrad of the Sitter space (let $\Phi = 1 - r^2$)

$$dS^2 = \Phi dt^2 - \frac{dr^2}{\Phi} - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

takes the form

$$\left[i\frac{\gamma^0}{\sqrt{\Phi}}\partial_t + i\sqrt{\Phi}\left(\gamma^3\partial_r + \frac{\gamma^1\sigma^{31} + \gamma^2 j^{32}}{r} + \frac{\Phi'}{2\Phi}\gamma^0\sigma^{03} \right) + \frac{1}{r}\Sigma_{\theta,\phi} - M \right]\Psi(x) = 0, \qquad (2)$$

where

$$\Sigma_{\theta,\phi} = i\gamma^1\partial_\theta + \gamma^2\frac{i\partial + i\sigma^{12}\cos\theta}{\sin\theta}.$$

With a simplifying substitution we get

$$\Psi(x) = r^{-1}\Phi^{-1/4}\psi(x),$$
$$\left(i\frac{\gamma^0}{\sqrt{\Phi}}\partial_t + i\sqrt{\Phi}\gamma^3\partial_r + \frac{1}{r}\Sigma_{\theta,\phi} - M \right)\psi = 0.$$

Below the spinor basis will be used.

2. Separation of the variables

Let us introduce a Dirac string potential in the de Sitter space-time model. It is convenient to start with the monopole Abelian potential in the Schwinger's form for the flat Minkowski space [41]

$$A^a(x) = \left(0, g\,\frac{(\vec{r}\times\vec{n})(\vec{r}\vec{n})}{r(r^2-(\vec{r}\vec{n})^2)} \right). \qquad (3)$$

Specifying $\vec{n} = (0,0,1)$ and translating the $A_\alpha(x)$ to the spherical coordinates, we get $A_\phi = g\cos\theta$. This potential A_ϕ obeys Maxwell equations in de Sitter space as well. Correspondingly, the Dirac equation in presence of this field A_ϕ takes the form

$$\left(i\frac{\gamma^0}{\sqrt{\Phi}}\partial_t + i\sqrt{\Phi}\gamma^3\partial_r + \frac{1}{r}\Sigma^k_{\theta,\phi} - M \right)\psi = 0,$$

where (below the notation $eg/\hbar c = k$ will be used)

$$\Sigma^k_{\theta,\phi} = i\gamma^1\partial_\theta + \gamma^2\,\frac{i\partial_\phi + (i\sigma^{12} - k)\cos\theta}{\sin\theta}. \qquad (4)$$

As readily verified, the wave operator in (4) commutes with the following three ones

$$J_1^k = l_1 + \frac{(i\sigma^{12}-k)\cos\phi}{\sin\theta},$$
$$J_2^k = l_2 + \frac{(i\sigma^{12}-k)\sin\phi}{\sin\theta},\; J_3^k = l_3 \qquad (5)$$

which in turn obey the $su(2)$ Lie algebra. Clearly, this monopole situation comes entirely under the Schrödinger [42], and Pauli [43] approach; detailed treatment of the method was given recently in [44]; similar technique when treating the problem of any spin particle in magnetic pole was used previously in [45], though with no connection with tetrad formalism.

Corresponding to diagonalization of the $\vec{J}_k^2$ and J_3^k, the function ψ is to be taken as ($D_\sigma \equiv D^j_{-m,\sigma}(\phi,\theta,0)$ stands for Wigner functions [46])

$$\psi^k_{\varepsilon jm}(t,r,\theta,\phi) = e^{-i\varepsilon t}\begin{vmatrix} f_1 D_{k-1/2} \\ f_2 D_{k+1/2} \\ f_3 D_{k-1/2} \\ f_4 D_{k+1/2} \end{vmatrix}. \qquad (6)$$

Further, with the use of recursive relations [46] we find how the $\Sigma^k_{\theta,\phi}$ acts on $\psi^k_{\varepsilon jm}$

$$\Sigma^k_{\theta,\phi}\psi^k_{\varepsilon jm} = i\sqrt{(j+1/2)^2 - k^2}\,e^{-i\varepsilon t} \times$$
$$\times \begin{vmatrix} -f_4 D_{k-1/2} (10) \\ +f_3; D_{k+1/2} (11) \\ +f_2 D_{k-1/2} (12) \\ -f_1 D_{k+1/2} \end{vmatrix}; \qquad (7)$$

hereafter the factor $\sqrt{(j+1/2)^2 - k^2}$ will be referred to as ν. For the $f_i(r)$, the radial system derived is

$$\frac{\varepsilon}{\sqrt{\Phi}} f_3 - i\sqrt{\Phi}\frac{d}{dr} f_3 - i\frac{\nu}{r} f_4 - M f_1 = 0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}} f_4 + i\sqrt{\Phi}\frac{d}{dr} f_4 + i\frac{\nu}{r} f_3 - M f_2 = 0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}} f_1 + i\sqrt{\Phi}\frac{d}{dr} f_1 + i\frac{\nu}{r} f_2 - M f_3 = 0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}} f_2 - i\sqrt{\Phi}\frac{d}{dr} f_2 - i\frac{\nu}{r} f_1 - M f_4 = 0. \qquad (8)$$

Else one operator can be diagonalized together with $i\partial_t, \vec{J}_k^2, J_3^k$: namely, a generalized Dirac operator

$$K^k = -i\gamma^0\gamma^3\Sigma^k_{\theta,\phi}. \qquad (9)$$

From the eigenvalue equation $K^k\psi_{\varepsilon jm} = \lambda\psi_{\varepsilon jm}$ we can produce two possible values for this λ and the corresponding restrictions on $f_i(r)$

$$\lambda = -\delta\sqrt{(j+1/2)^2 - k^2},$$
$$f_4 = \delta f_1,\; f_3 = \delta f_2. \qquad (10)$$

Correspondingly, the system (8) reduces to

$$\left(\sqrt{\Phi}\frac{d}{dr}+\frac{\nu}{r}\right)f+\left(\frac{\varepsilon}{\sqrt{\Phi}}+\delta M\right)g=0,$$
$$\left(\sqrt{\Phi}\frac{d}{dr}-\frac{\nu}{r}\right)g-\left(\frac{\varepsilon}{\sqrt{\Phi}}-\delta M\right)f=0, \quad (11)$$

we have translated equations to new functions $f=(f_1+f_2)/\sqrt{2},\ g=(f_1-f_2)i\sqrt{2}$.
Note the quantization rules:

$$\frac{eg}{hc}=\pm 1/2, \pm 1, \pm 3/2, \ldots;$$
$$j=|k|-1/2, |k+1/2, |k|+3/2, \ldots \quad (12)$$

The case of minimal value $j_{\min}=|k|-1/2$ must be separated and treated in a special way:

$$\psi^{k>0}_{j_{\min.}}(x)=e^{-i\varepsilon t}\left|\begin{array}{c} f_1(r)D_{k-1/2} \\ 0(20) \\ f_3(r)D_{k-1/2} \\ 0 \end{array}\right|;$$

$$\psi^{k<0}_{j_{\min}}(x)=e^{-i\varepsilon t}\left|\begin{array}{c} 0 \\ f_2(r)D_{k+1/2} \\ 0 \\ f_4(r)D_{k+1/2} \end{array}\right|, \quad (13)$$

and the relation $\Sigma_{\theta,\phi}\Psi_{j_{\min}}=0$ holds. Thus, at every k, the $j_{\min}$-equation has the same form

$$\left(i\frac{\gamma^0}{\sqrt{\Phi}}\partial_t+i\gamma^3\sqrt{\Phi}(\partial_r+\frac{1}{r})-M\right)\psi_{j_{\min}}=0;$$

which leads to the same radial system

$$k=+1/2, +1, \ldots$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_3-i\sqrt{\Phi}\frac{d}{dr}f_3-Mf_1=0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_1+i\sqrt{\Phi}\frac{d}{dr}f_1-Mf_3=0; \quad (14)$$
$$k=-1/2, -1, \ldots$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_4+i\sqrt{\Phi}\frac{d}{dr}f_4-Mf_2=0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_2-i\sqrt{\Phi}\frac{d}{dr}f_2-Mf_4=0. \quad (15)$$

3. Radial equations in the case $j_{\min}$

For brevity, let us examine only the case of the minimal value of j:

$$k=+1/2, +1, \ldots$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_3-i\sqrt{\Phi}\frac{d}{dr}f_3-Mf_1=0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_1+i\sqrt{\Phi}\frac{d}{dr}f_1-Mf_3=0; \quad (16)$$

from whence for new functions $h=(f_1+f_3)/\sqrt{2},\ g=(f_1-f_3)/i\sqrt{2}$ we derive $k=+1/2, +1, \ldots$

$$\sqrt{\Phi}\frac{d}{dr}h+\left(\frac{\varepsilon}{\sqrt{\Phi}}+M\right)g=0,$$
$$\sqrt{\Phi}\frac{d}{dr}g-\left(\frac{\varepsilon}{\sqrt{\Phi}}-M\right)h=0. \quad (17)$$

In the same manner for another case we have

$$k=-1/2, -1, \ldots$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_4+i\sqrt{\Phi}\frac{d}{dr}f_4-Mf_2=0,$$
$$\frac{\varepsilon}{\sqrt{\Phi}}f_2-i\sqrt{\Phi}\frac{d}{dr}f_2-Mf_4=0; \quad (18)$$

where $g=(f_2+f_4)/\sqrt{2},\ h=(f_2-f_4)/i\sqrt{2}$ we obtain

$$\sqrt{\Phi}\frac{d}{dr}h+\left(\frac{\varepsilon}{\sqrt{\Phi}}-M\right)g=0,$$
$$\sqrt{\Phi}\frac{d}{dr}g-\left(\frac{\varepsilon}{\sqrt{\Phi}}+M\right)h=0. \quad (19)$$

To exclude additional non-physical singular points, let us perform special transformation over the functions

$$g+h=e^{-i\rho/2}(F+G),$$
$$g-h=e^{+i\rho/2}(F-G). \quad (20)$$

After simple calculation we arrive at

$$k=+1/2, +3/2, \ldots$$
$$\left(\frac{d}{d\rho}-i\varepsilon\frac{\sin\rho}{\cos\rho}\right)F+$$
$$+\left(+\varepsilon+M-\frac{i}{2}\right)G=0,$$
$$\left(\frac{d}{d\rho}+i\varepsilon\frac{\sin\rho}{\cos\rho}\right)G+$$
$$+\left(-\varepsilon+M-\frac{i}{2}\right)F=0; \quad (21)$$

$k=-1/2,-3/2,...$

$$\left(\frac{d}{d\rho}-i\varepsilon\frac{\sin\rho}{\cos\rho}\right)G+ +\left(+\varepsilon-M-\frac{i}{2}\right)H=0,$$
$$\left(\frac{d}{d\rho}+i\varepsilon\frac{\sin\rho}{\cos\rho}\right)H+ +\left(-\varepsilon-M-\frac{i}{2}\right)G=0. \quad (22)$$

The difference between (21) and (22) consists in the only change $M \longleftrightarrow -M$.

In particular, the system (21) being translated to the variable z

$$\sin\rho=\sqrt{z},\frac{d}{d\rho}=2\sqrt{z(1-z)}\,\frac{d}{dz},$$

will take the form

$$\sqrt{z(1-z)}\left(\frac{d}{dz}-\frac{i\varepsilon/2}{1-z}\right)F+ +\frac{M+\varepsilon-i/2}{2}G=0,$$
$$\sqrt{z(1-z)}\left(\frac{d}{dz}+\frac{i\varepsilon/2}{1-z}\right)G+ +\frac{M-\varepsilon-i/2}{2}F=0. \quad (23)$$

From (23) it follow 2-nd order differential equations for F and G respectively

$$z(1-z)\frac{d^2F}{dz^2}+(\frac{1}{2}-z)\frac{dF}{dz}+ +\left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2+\frac{\varepsilon(\varepsilon-i)}{4(1-z)}\right]F=0,$$
$$z(1-z)\frac{d^2G}{dz^2}+(\frac{1}{2}-z)\frac{dG}{dz}+ +\left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2+\frac{\varepsilon(\varepsilon+i)}{4(1-z)}\right]G=0.$$

Let us introduce substitutions

$$F=z^A(1-z)^B\bar{F}(z),\ \ G=z^K(1-z)^L\bar{G}(z).$$

At A and B taken accordingly

$$A=\frac{1}{2},0, \qquad B=-\frac{i\varepsilon}{2},\frac{1+i\varepsilon}{2} \quad (24)$$

equation for $\bar{F}$ looks

$$z(1-z)\frac{d^2\bar{F}}{dz^2}+ +[2A+\frac{1}{2}-(2A+2B+1)z]\frac{d\bar{F}}{dz}+ +[-\frac{1}{4}(M-\frac{i}{2})^2-(A+B)^2]\bar{F}=0$$

which is of hypergeometric type with parameters

$$a=A+B+\frac{iM+1/2}{2},$$
$$b=A+B-\frac{iM+1/2}{2},\ \ c=2A+1/2.$$

To have solutions non-vanishing in the origin $z=0$, we take zero $A=0$. Thus there arise two sorts of solutions depending on a chosen B (in each case two linearly independent solutions, regular and singular in the origin, are written down)

the first

$$c=+1/2, \qquad \bar{F}^{(1)}_{non-zero}(z)=F(a,b,c;z),$$
$$\bar{F}^{(1)}_{zero}=z^{1-c}F(a+1-c,b+1-c,2-c;z),$$
$$a=\frac{-i\varepsilon}{2}+\frac{iM+1/2}{2},\ \ b=\frac{-i\varepsilon}{2}-\frac{iM+1/2}{2};$$

the second

$$\gamma=+1/2,\bar{F}^{(2)}_{non-zero}=F(\alpha,\beta,\gamma;z),\bar{F}^{(2)}_{zero}= =z^{1-\gamma}F(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z),$$

where

$$\alpha=\frac{1+i\varepsilon}{2}+\frac{iM+1/2}{2},$$
$$\beta=\frac{1+i\varepsilon}{2}-\frac{iM+1/2}{2}.$$

Now let us turn to equation for $\bar{G}$; at K and L chosen according to

$$K=\frac{1}{2},0, \qquad L=\frac{i\varepsilon}{2},\frac{1-i\varepsilon}{2} \quad (25)$$

it looks

$$z(1-z)\frac{d^2\bar{G}}{dz^2}+ \quad (36)$$
$$+\left[2K+\frac{1}{2}-(2K+2L+1)z\right]\frac{d\bar{G}}{dz}+ +\left[-\frac{1}{4}\left(M-\frac{i}{2}\right)^2-(K+L)^2\right]\bar{G}=0,$$

which is of hypergeometric type

$$a' = K + L + \frac{iM + 1/2}{2},$$
$$b' = K + L - \frac{iM + 1/2}{2}, \quad c' = 2K + \frac{1}{2}.$$

We start with solutions non-vanishing in the origin $z = 0$, we take zero $K = 0$. Thus there arise two sorts of solutions depending on a chosen B (in each case two linearly independent solutions, regular and singular in the origin, are written down)

the first

$$c' = +1/2, \quad \bar{G}^{(1)}_{non-zero} = F(a', b', c'; z), \bar{G}^{(1)}_{zero} =$$
$$= z^{1-c'} F(a' + 1 - c', b' + 1 - c', 2 - c'; z),$$
$$a' = \frac{+i\varepsilon}{2} + \frac{iM + 1/2}{2}, \quad b' = \frac{+i\varepsilon}{2} - \frac{iM + 1/2}{2};$$

the second

$$\gamma' = +1/2, \quad \bar{G}^{(2)}_{non-zero}(z) = F(\alpha', \beta', \gamma'; z),$$
$$\bar{G}^{(2)}_{zero} = z^{1-\gamma'} F(\alpha' + 1 - \gamma', \beta' + 1 - \gamma',$$
$$2 - \gamma'; z), \qquad \alpha' = \frac{1 - i\varepsilon}{2} + \frac{iM + 1/2}{2},$$
$$\beta' = \frac{1 - i\varepsilon}{2} - \frac{iM + 1/2}{2}.$$

Thus, we have constructed the following four regular solutions

$$F^{(1)}_{non-zero}, G^{(1)}_{non-zero}, G^{(2)}_{non-zero}.$$

Due to the known identity for hypergeometric functions

$$F(A, B, C; z) =$$
$$= (1 - z)^{C-A-B} F(C - A, C - B, C; z)$$

we readily conclude that there exist only two different ones

$$F^{(1)}_{non-zero} = F^{(2)}_{non-zero},$$
$$G^{(1)}_{non-zero} = G^{(2)}_{non-zero}. \qquad (26)$$

The same is true for zero-solutions

$$F^{(1)}_{zero} = F^{(2)}_{zero}, \qquad G^{(1)}_{zero} = G^{(1)}_{zero}. \qquad (27)$$

Assuming relationship

$$2\sqrt{z(1-z)} \left(\frac{d}{dz} - \frac{i\varepsilon/2}{1-z} \right) F^{(1)}_{non-zero} +$$
$$+ (M + \varepsilon - i/2) G^{(2)}_{zero} = 0.$$

we readily derive

$$a F_0^{non-zero} + ic G_0^{zero} = 0. \qquad (28)$$

And similarly, we can expect

$$2\sqrt{z(1-z)} \left(\frac{d}{dz} + \frac{i\varepsilon/2}{1-z} \right) G^{(1)}_{non-zero} +$$
$$+ (M - \varepsilon - i/2) F^{(2)}_{zero} = 0,$$

so that

$$a' G_0^{non-zero} + ic' F_0^{zero} = 0. \qquad (29)$$

4. Standing and running waves, $j = j_{\min}$

Let write down results we need to proceed further

$$F_{non-zero} = (1 - z)^{-i\varepsilon/2} U_1,$$
$$F_{zero} = (1 - z)^{-i\varepsilon/2} U_5,$$
$$G_{non-zero} = (1 - z)^{+i\varepsilon/2} V_1,$$
$$G_{zero} = (1 - z)^{+i\varepsilon/2} V_5, \qquad (30)$$

$$U_1 = F(a, b, c, z), \qquad V_1 = F(a', b', c', z),$$
$$a = \frac{-i\varepsilon}{2} + \frac{iM + 1/2}{2},$$
$$b = \frac{-i\varepsilon}{2} - \frac{iM + 1/2}{2}, \qquad c = 1/2,$$
$$a' = \frac{+i\varepsilon}{2} + \frac{iM + 1/2}{2},$$
$$b' = \frac{+i\varepsilon}{2} - \frac{iM + 1/2}{2}, \qquad c' = 1/2.$$

We readily derive

$$F_{non-zero} = (1 - z)^{-i\varepsilon/2} U_1$$
$$= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F_{out} +$$
$$+ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F_{in},$$
$$F_{zero} = (1 - z)^{-i\varepsilon/2} U_5$$
$$= \frac{\Gamma(2 - c)\Gamma(c - a - b)}{\Gamma(1 - a)\Gamma(1 - b)} F_{out} +$$
$$+ \frac{\Gamma(2 - c)\Gamma(a + b - c)}{\Gamma(a + 1 - c)\Gamma(b + 1 - c)} F_{in},$$

where

$$F_{out}=(1-z)^{-i\varepsilon/2}U_2, F_{in}=(1-z)^{-i\varepsilon/2}U_6.$$

$$G_{non-zero}=(1-z)^{+i\varepsilon/2}V_1$$

$$=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}G_{in}+$$

$$+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}G_{out},$$

$$G_{zero}=(1-z)^{+i\varepsilon/2}V_5$$

$$=\frac{\Gamma(2-c')\Gamma(c'-a'-b')}{\Gamma(1-a')\Gamma(1-b')}G_{in}+$$

$$+\frac{\Gamma(2-c')\Gamma(a'+b'-c')}{\Gamma(a'+1-c')\Gamma(b'+1-c')}G_{out},$$

where

$$G_{in}=(1-z)^{+i\varepsilon/2}V_2, G_{out}=(1-z)^{+i\varepsilon/2}V_6.$$

5. Discussion and conclusions

To understand better the situation, let us consider the case of minimal $j_{\min}$ in the limit of vanishing curvature. It is convenient to start with the first order systems for minimal values $j_{\min}$ in the case of Minkowski space:

$k=+1/2,+1,\ldots$

$$\varepsilon f_3 - i\frac{d}{dr}f_3 - Mf_1=0,$$

$$\varepsilon f_1 + i\frac{d}{dr}f_1 - Mf_3=0; \qquad (31)$$

$k=-1/2,-1,\ldots$

$$\varepsilon f_4 + i\frac{d}{dr}f_4 - Mf_2=0,$$

$$\varepsilon f_2 - i\frac{d}{dr}f_2 - Mf_4=0. \qquad (32)$$

Let us detail the case of positive $k=+1/2,+1,\ldots$. With notation

$$\frac{f_1+f_3}{\sqrt{2}}=h(r), \frac{f_1-f_3}{i\sqrt{2}}=g(r); \qquad (33)$$

relevant equations are

$$\frac{d}{dr}h+(\varepsilon+M)g=0,$$

$$\frac{d}{dr}g-(\varepsilon-M)h=0. \qquad (34)$$

Further, with the substitutions

$$h(r)=He^{\gamma r}, \qquad g(r)=Ge^{\gamma r} \qquad (35)$$

we get (first let it be $(\varepsilon^2-M^2)>0$)

$$\gamma^2=-(\varepsilon^2-M^2)=-p^2,$$

$$\gamma=+ip,-ip, G\gamma-(\varepsilon-M)H=0. \qquad (36)$$

Thus we have two linearly independent solutions

$$h_1(r)=H_1e^{+ipr}, g_1(r)=G_1e^{+ipr},$$

$$G_1=\frac{\varepsilon-M}{ip}H_1 \qquad (37)$$

and

$$h_2(r)=H_2e^{-ipr}, g_2(r)=G_2e^{-ipr},$$

$$G_2=\frac{\varepsilon-M}{-ip}H_2; \qquad (38)$$

for simplicity, we will take $H_1=H_2=1$. It is convenient to introduce linear combinations of these solutions

$$\frac{h_1(r)+h_2(r)}{2}=\cos pr,$$

$$\frac{g_1(r)+g_2(r)}{2}=\frac{\varepsilon-M}{p}\sin pr; \qquad (39)$$

$$\frac{h_1(r)-h_2(r)}{2i}=\sin pr,$$

$$\frac{g_1(r)-g_2(r)}{2i}=\frac{\varepsilon-M}{-p}\cos pr. \qquad (40)$$

Now let us specify the case $(\varepsilon^2-M^2)<0$:

$$\gamma^2=-(\varepsilon^2-M^2)\equiv+q^2,$$

$$\gamma=+q,-q,$$

$$G\gamma-(\varepsilon-M)H=0. \qquad (41)$$

We have two linearly independent solutions

$$h_1(r)=H_1e^{+qr}, g_1(r)=G_1e^{+qr},$$

$$G_1=\frac{\varepsilon-M}{q}H_1; \qquad (42)$$

$$h_2(r)=H_2e^{-qr}, g_2(r)=G_2e^{-qr},$$

$$G_2 = \frac{\varepsilon - M}{-q} H_2. \qquad (43)$$

Below $H_1 = H_2 = 1$. We can introduce two linear combinations of these solutions

$$\frac{h_1(r)+h_2(r)}{2} = \cosh qr,$$

$$\frac{g_1(r)+g_2(r)}{2} = \frac{\varepsilon - M}{q} \sinh qr, \qquad (44)$$

$$\frac{h_1(r)-h_2(r)}{2} = \sinh qr,$$

$$\frac{g_1(r)-g_2(r)}{2} = \frac{\varepsilon - M}{q} \cosh qr. \qquad (45)$$

Evidently, above constructed solutions in de Sitter model provide us with generalizations of these in Minkowski space. It may be verified additionally by direct limiting process when $\rho \to \infty$. To this end, let us translate solutions in de Sitter space to usual units (ρ is the curvature radius, E is the energy, c is the light velocity, m is the electron mass)

$$F_{non-zero} = \left(1 - \frac{R^2}{\rho^2}\right)^{-i\frac{E\rho}{2c\hbar}} F(a,b,c;\frac{R^2}{\rho^2}),$$

$$F_{zero} = R\left(1 - \frac{R^2}{\rho^2}\right)^{+i\frac{E\rho}{2c\hbar}+1/2} \times$$

$$\times F(a+1-c, b+1-c, 2-c; \frac{R^2}{\rho^2}),$$

$$G_{non-zero} = \left(1 - \frac{R^2}{\rho^2}\right)^{+i\frac{E\rho}{2c\hbar}} F(a',b',c;\frac{R^2}{\rho^2}),$$

$$G_{zero} = R\left(1 - \frac{R^2}{\rho^2}\right)^{-i\frac{E\rho}{2c\hbar}+1/2} \times$$

$$\times F(a'+1-c, b'+1-c, 2-c; \frac{R^2}{\rho^2}),$$

Parameters of hypergeometric functions are given by

$$c = \frac{1}{2}, a = \frac{1}{2}\left[+1/2 + i(\frac{mc\rho}{\hbar} - \frac{E\rho}{c\hbar})\right],$$

$$b = \frac{1}{2}\left[-i(\frac{mc\rho}{\hbar} + \frac{E\rho}{c\hbar}) - 1/2\right],$$

$$c' = \frac{1}{2}, a' = \frac{1}{2}\left[+1/2 + i(\frac{mc\rho}{\hbar} + \frac{E\rho}{c\hbar})\right],$$

$$b' = \frac{1}{2}\left[-i(\frac{mc\rho}{\hbar} - \frac{E\rho}{c\hbar}) - 1/2\right].$$

Let us examine the limiting procedure at $\rho \to \infty$ in $F(a,b,c;R^2/\rho^2)$. Because

$$\frac{1}{1!}\frac{ab}{c}\frac{R^2}{\rho^2} \to -\frac{1}{2!}(pR)^2,$$

$$\frac{1}{2!}\frac{a(a+1)b(b+1)}{c(c+1)}\frac{R^2}{\rho^2} \to +\frac{(pR)^4}{4!},$$

$$\frac{1}{3!}\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{R^2}{\rho^2}$$

$$\to -\frac{(pR)^6}{6!},$$

and so on, we obtain the limiting relations

$$\lim_{\rho\infty} F(a,b,c;\frac{R^2}{\rho^2}) = \cos pr \qquad \Rightarrow$$

$$\lim_{\rho\infty} F_{non-zero}(R) = \cos pr,$$

$$\lim_{\rho\infty} G_{non-zero}(R) = \cos pr. \qquad (46)$$

In the same manner, let us examine the function

$$RF(a+1-c, b+1-c, 2-c; \frac{R^2}{\rho^2}),$$

$$A = a+1-c = \frac{3/2 + i(M+\varepsilon)}{2},$$

$$B = b+1-c = \frac{1/2 - i(M-\varepsilon)}{2}, C = 3/2.$$

Taking into account

$$\frac{AB}{C} \to -\frac{1}{3!}(pR)^2,$$

$$\frac{1}{2!}\frac{A(A+1)B(B+1)}{C(C+1)} \to +\frac{1}{5!}(pR)^4,$$

$$\frac{1}{3!}\frac{A(A+1)(A+2)B(B+1)(B+2)}{C(C+1)(C+2)}$$

$$\to -\frac{1}{7!}(pR)^6,$$

and so on, we arrive at the relationships

$$\lim_{\rho\infty} pRF(a+1-c, b+1-c, 2-c; \frac{R^2}{\rho^2}) = \sin pR,$$

$$\lim_{\rho\infty} pRF_{zero} = \sin pR, \lim_{\rho\infty} pRG_{zero} = \sin pR.$$

Thus, indeed, solutions in de Sitter model are extensions of more simple and well-known ones in Minkowski space.

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ЧАСТИНКИ ЗІ СПІНОМ 1/2 У ПРИСУТНОСТІ АБЕЛЕВА МОНОПОЛЯ НА ФОНІ ПРОСТОРУ-ЧАСУ ДЕ СІТТЕРА

У космологічній моделі де Сіттера визначено потенціал діраківського монополя. Досліджено рівняння Дірака в присутності цього поля в статичних координатах простору де Сіттера. Замість спінорних монопольних гармонік використано техніку *D*-функцій Вігнера. Після розділення змінних проведено детальний аналіз отриманих радіальних рівнянь, в термінах гіпергеометричних функцій побудовано чотири типи розв'язків: сингулярне, регулярне, біжучі хвилі, які сходяться і розбігаються. Знайдено повний набір хвильових розв'язків $\Psi_{\varepsilon,j,m,\lambda}(t,r,\theta,\phi)$, особливу увагу приділено дослідженню станів з мінімальним значенням повного кутового моменту j_{min}.

Ключові слова: потенціал діраківського монополя, простір-час де Сіттера, рівняння Дірака, *D*-функція Вігнера, гіпергеометричні функції.

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ЧАСТИЦЫ СО СПИНОМ 1/2 В ПРИСУТСТВИИ АБЕЛЕВА МОНОПОЛЯ НА ФОНЕ ПРОСТРАНСТВА-ВРЕМЕНИ ДЕ СИТТЕРА

В космологической модели де Ситтера определен потенциал дираковского монополя. Уравнение Дирака в присутствии этого поля исследуется в статических координатах пространства де Ситтера. Вместо спинорных монопольных гармоник используется техника *D*-функций Вигнера. После разделения переменных проведен детальный анализ полученных радиальных уравнений, четыре типа решений: сингулярное, регулярное, расходящиеся и сходящиеся бегущие волны построены в терминах гипергеометрических функций. Найден полный набор волновых решений $\Psi_{\varepsilon,j,m,\lambda}(t,r,\theta,\phi)$, особое внимание уделено исследованию состояний с минимальным значением полного углового момента j_{min}.

Ключевые слова: потенциал дираковского монополя, пространство-время де Ситтера, уравнение Дирака, *D*-функция Вигнера, гипергеометрическая функция.