УДК 539.12

E. Ovsiyuk
Mozyr State Pedagogical University
e-mail: *e.ovsiyuk@mail.ru*

GENERALIZED PLANE WAVES FOR SCHRÖDINGER AND DIRAC PARTICLES ON THE BACKGROUND OF BOLYAI–LOBACHEVSKY GEOMETRY: SIMULATING OF A SPECIAL MEDIUM

Bolyai–Lobachevsky geometry substantially affects quantum-mechanical particles, simulating a medium with special reflecting properties of an ideal mirror.

For Scrödinger particle the problem reduces to a second order differential equation which can be associated with one-dimensional Schrödinger problem for a particle in external potential field $U(z) = U_0 e^{2z}$. In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient $R = 1$. Hyperbolic geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the particle into the effective medium, depends on the parameters of quantum solutions $\varepsilon,\ k_1^2 + k_2^2$, and the curvature radius ρ. Similar analysis is performed for the case of a Dirac spin 1/2 particle; additional to the quantum numbers $\varepsilon,\ k_1^2 + k_2^2$ for the spin 0 particle here is a quantum number related with an extended helicity operator.

Key words: geometry Bolyai–Lobachevsky, plane wave, Schrödinger equation, Dirac equation, hypergeometric function.

Introduction

It is known that in the field theory of elementary particles, the basis of plane waves is of the most use. However, in presence of a curvature, any common plane wave solutions do not exist. Therefore, of a special interest are examples non-Euclidean spaces in which some analogues of such solutions can be constructed. In the paper [1], it was shown that in the Lobachevsky space there are such solutions for particles with spin 0; also see the books by Gelfand–Graev–Vilenkin [2], [3]. An analog of plane waves in a space of constant positive curvature was studied by Volobuev [4]. The later treatment of this problem was given in [5]. Solutions of the plane wave type for Maxwell's equations have been considered in [6]–[9]. In [10], the problem of constructing solutions of the Dirac equation in the Lobachevsky space was studied on the base of the method of squaring; in particular, it was pointed out the possibility of constructing solutions of the Dirac plane waves starting with Shapiro's scalar waves. In this paper we will construct a complete basis of solutions of the plane wave type for Dirac and Weyl particles in the Lobachevsky space, applying the method of separation of the variables in a special system of quasi-cartesian coordinates closely related to horospherical coordinates.

To understand the physical meaning of the system under consideration, it should be mentioned that Lobachevsky geometry simulates a medium with special constitutive relations. The situation being specified in quasi-cartesian coordinates (x, y, z) was treated in [9]. Exact solutions of the Maxwell equations in complex 3-vector form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field $U(z) = U_0 e^{2z}$. In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient $R = 1$; in electrodynamics context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency ω, $k_1^2+k_2^2$, and the curvature radius ρ.

In the present paper, that analysis will be extended to the case of particles with spin $1/2$, described by equations of Dirac and Weyl. The generalized spinor plane waves can find application in the analysis of the behavior of fermions particles on cosmological scales, or in simulating special media affecting the spinor particles.

1. On the solutions of the Schrödinger equation

In the Lobachevsky space–time parameterized by quasi-cartesian coordinates

$$dS^2 = dt^2 - e^{-2z}(dx^2+dy^2) - dz^2;$$

the element of volume given by dV and the sign of the z are substantial, in particular when referring to the probabilistic interpretation of the wave functions

$$dW = |\Psi|^2\, dV = |\Psi|^2\, e^{-2z} dx\, dy\, dz.$$

Let us describe some details of the parametrization of the space by coordinates (x, y, z). It is known that this model can be identified with a branch of hyperboloid in 4-dimension flat space

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2, u_0 = +\sqrt{\rho^2+\mathbf{u}^2}.$$

Coordinates x, y, z are referred to u_a by relations

$$u_1 = xe^{-z}, u_2 = ye^{-z},$$

$$u_3 = \frac{1}{2}[(e^z - e^{-z}) + (x^2+y^2)e^{-z}],$$

$$u_0 = \frac{1}{2}[(e^z + e^{-z}) + (x^2+y^2)e^{-z}]. \quad (1.1a)$$

It is convenient to employ 3-dimensional Poincaré realization for Lobachevsky space as an inside part of 3-sphere:

$$q_i = \frac{u_i}{\sqrt{\rho^2+u_1^2+u_2^2+u_3^2}}, q_i q_i < +1. \quad (1.1b)$$

Quasi-cartesian coordinates (x, y, z) are referred to q_i as follows

$$x = \frac{q_1}{1-q_3}, y = \frac{q_2}{1-q_3}, e^z = \frac{\sqrt{1-q^2}}{1-q_3}. \quad (1.1c)$$

In particular, note that on the axis $q_1 = 0, q_2 = 0, q \in (-1,+1)$ relations $(1.1c)$ assume the form

$$q_3 \longrightarrow +1, e^z \longrightarrow +\infty, z \longrightarrow +\infty;$$
$$q_3 \longrightarrow -1, e^z \longrightarrow +0, z \longrightarrow -\infty. \quad (1.2)$$

Schrödinger equation in Riemannian space [11], in quasi-cartesian coordinates $(1.1a)$ takes the form

$$i\hbar\frac{\partial}{\partial t}\Psi = -\frac{\hbar^2}{2M}\left(e^{2z}\frac{\partial^2}{\partial x^2} + e^{2z}\frac{\partial^2}{\partial y^2} + \right.$$
$$\left. + e^{2z}\frac{\partial}{\partial z}e^{-2z}\frac{\partial}{\partial z}\right)\Psi.$$

The variables are separated by the substitution $\Psi = e^{-iEt/\hbar}e^{ik_1x}\, e^{ik_2y}\, f(z)$:

$$\left(\frac{d^2}{dz^2} - 2\frac{d}{dz} + \varepsilon - e^{2z}(k_1^2+k_2^2)\right) f(z) = 0, \quad (1.3a)$$

where a dimensionless quantity used $\varepsilon = 2ME\rho^2/\hbar^2$, ρ – curvature radius of the space. Elementary substitution $f = e^z\varphi(z)$ in equation $(1.3a)$ gives a Schrödinger-like equation

$$\left(\frac{d^2}{dz^2} + \varepsilon - 1 - (k_1^2+k_2^2)e^{2z}\right)\varphi(z) = 0 \quad (1.3b)$$

with potential function

$$U(z) = 1 + (k_1^2+k_2^2)e^{2z}. \quad (1.3c)$$

Note that the probabilistic interpretation of the wave function after the transformation to φ reads

$$dW = |\Psi|^2\, dV = |\varphi|^2\, dxdydz. \quad (1.4)$$

An easily interpretable physical solution for $\varepsilon > 1$ is the following: on the left we have the superposition of two waves, falling from the left and reflected. On the right behind the barrier, the wave function must sharply decrease to zero.

It should be noted that the case $k_1 = 0,\ k_2 = 0$ is special: the equation $(1.3a)$ is very much changed because the potential function disappears

$$\left(\frac{d^2}{dz^2} - 2\frac{d}{dz} + \varepsilon \right) f(z) = 0,$$

$$f = e^{(1\pm i\sqrt{\varepsilon-1})z},\ \varphi = e^{(\pm i\sqrt{\varepsilon-1})z}, \qquad (1.5)$$

and the function φ is a solutions of the type of ordinary plane wave.

Let us turn to the general case and in eq. $(1.3a)$ introduce the variable

$$\sqrt{k_1^2 + k_2^2}e^z = Z, Z \in (0,+\infty);$$

the equation takes the form

$$\left(\frac{d^2}{dZ^2} - \frac{1}{Z}\frac{d}{dZ} + \frac{\varepsilon}{Z^2} - 1 \right) f(Z) = 0; \qquad (1.6)$$

with the help of a substitution $f = \sqrt{Z}F$, one can remove the term with the first derivative

$$\left(\frac{d^2}{dZ^2} + \frac{\varepsilon - 3/4}{Z^2} - 1 \right) F(Z) = 0.$$

This form makes it easy to find the asymptotical behavior of solutions

$$(z \longrightarrow -\infty) Z \to 0, F \sim Z^{1/2\pm i\sqrt{\varepsilon-1}},$$
$$f \sim Z^{1\pm i\sqrt{\varepsilon-1}}, \varphi \sim e^{\pm i\sqrt{\varepsilon-1}z}; \qquad (1.7)$$

$$(z \longrightarrow +\infty) Z \to +\infty, F \sim e^{\pm Z},$$
$$f = \sqrt{Z}e^{\pm Z}, \varphi \sim e^{-z/2}\exp(\pm\sqrt{k_1^2 + k_2^2}e^z). \quad (1.8)$$

We now turn to the construction of exact solutions of (1.6) in the entire range of coordinate z. We seek solutions in the form of $f(Z) = Z^A e^{BZ} F(Z)$; at A, B chosen according (for definiteness, we take the minus sign before the root in the expression for A; assuming $\varepsilon > 1$)

$$A = 1 - i\sqrt{\varepsilon - 1}, \qquad B^2 = 1, \qquad (1.9)$$

the equation (1.6) gives (let us make another change $Z = y/2$ and let $B = -1$) an equation for the confluent hypergeometric function

$$y\frac{d^2Y}{dZ^2} + (c - y)\frac{dY}{dy} - aY = 0,$$
$$c = 2a, a = A - 1/2 = 1/2 - i\sqrt{\varepsilon - 1},$$
$$f(Z) = y^{a+1/2}e^{-y/2}Y(y). \qquad (1.10)$$

We use two pairs of linearly independent solutions [12]

$$Y_1 = \Phi(a, 2a, y), Y_2 = y^{1-2a}\Phi(1-a, 2-2a, y);$$
$$Y_5 = \Psi(a, 2a, y), Y_7 = e^y\Psi(a, 2a, -y). \qquad (1.11)$$

These pairs of solutions are related by Kummer linear relations [12]

$$Y_5 = \frac{\Gamma(1-2a)}{\Gamma(1-a)}Y_1 + \frac{\Gamma(2a-1)}{\Gamma(a)}Y_2,$$
$$Y_7 = \frac{\Gamma(1-2a)}{\Gamma(1-a)}Y_1 - \frac{\Gamma(2a-1)}{\Gamma(a)}Y_2, \quad (1.12a)$$

which after multiplication by $y^{a+1/2}e^{-y/2}$ take the form

$$f_5 = \frac{\Gamma(1-2a)}{\Gamma(1-a)}f_1 + \frac{\Gamma(2a-1)}{\Gamma(a)}f_2,$$
$$f_7 = \frac{\Gamma(1-2a)}{\Gamma(1-a)}f_1 - \frac{\Gamma(2a-1)}{\Gamma(a)}f_2. \quad (1.12b)$$

Note that the solutions Y_1 and Y_2 describe the waves with asymptotic behavior $z \to -\infty, (y \longrightarrow 0)$

$$f_1 \sim \left(2\sqrt{k_1^2 + k_2^2}\right)^{1-i\sqrt{\varepsilon-1}} e^z e^{-i\sqrt{\varepsilon-1}z},$$
$$f_2 \sim \left(2\sqrt{k_1^2 + k_2^2}\right)^{1+i\sqrt{\varepsilon-1}} e^z e^{+i\sqrt{\varepsilon-1}z}. \quad (1.13)$$

Thus, for example, the function Y_5 (and the related φ_5) at negative $z \longrightarrow -\infty$ behaves as a superposition of two plane waves according to

$$\varphi_5 \sim \frac{\Gamma(1-2a)}{\Gamma(1-a)}(2\sqrt{k_1^2 + k_2^2})^{1-i\sqrt{\varepsilon-1}}e^{-i\sqrt{\varepsilon-1}z}$$
$$+\frac{\Gamma(2a-1)}{\Gamma(a)}(2\sqrt{k_1^2 + k_2^2})^{1+i\sqrt{\varepsilon-1}}e^{+i\sqrt{\varepsilon-1}z}. \quad (1.14)$$

We define the reflection coefficient as the square modulus of the amplitude ratio in a superposition of plane waves

$$M_- e^{-i\sqrt{\varepsilon-1}z} + M_+ e^{+i\sqrt{\varepsilon-1}z}, R = \left|\frac{M_-}{M_+}\right|^2,$$

$$R = \left|\frac{\Gamma(1-2a)}{\Gamma(2a-1)}\frac{\Gamma(a)}{\Gamma(1-a)}\right|^2. \qquad (1.15a)$$

We take into account

$$1-2a = +2i\sqrt{\varepsilon-1}, 2a-1 = -2i\sqrt{\varepsilon+1},$$

$$a = 1/2 - i\sqrt{\varepsilon-1}, 1-a = 1/2 + i\sqrt{\varepsilon-1},$$

then

$$R = \left|\frac{\Gamma(+2i\sqrt{\varepsilon-1})}{\Gamma(-2i\sqrt{\varepsilon-1})}\right|^2 \times$$

$$\times\left|\frac{\Gamma(1/2 - i\sqrt{\varepsilon-1})}{\Gamma(1/2 + i\sqrt{\varepsilon-1})}\right|^2 \equiv 1. \qquad (1.15b)$$

We find the behavior of Y_5 at large y. Using the known asymptotic relation [12]

$$Y_5 = \Psi(a,c,y) \sim y^{-a},$$

we get at $z \to +\infty$:

$$f_5 = y^{a+1/2} e^{-y/2}\, Y_5 \sim y^{1/2} e^{-y/2} \sim$$

$$\sim (2\sqrt{k_1^2 + k_2^2 e^z})^{1/2} \exp(-\sqrt{k_1^2 + k_2^2 e^z})$$

$$\longrightarrow \exp^{-e^{+\infty}} = 0. \qquad (1.16)$$

Thus, the solution f_5 describes the expected situation: wave going from the left is reflected with probability 1 on the effective barrier; behind the barrier the solutions sharply decrease to zero. It is easy to find the critical point, after which wave function must sharply decrease

$$\varepsilon - 1 = (k_1^2 + k_2^2)e^{2z} \Rightarrow z_0 = \ln\sqrt{\frac{\varepsilon-1}{k_1^2+k_2^2}},$$

in the usual units, this critical point is described by the relation

$$z_0 = \rho \ln\sqrt{\frac{2mE\rho^2/\hbar^2 - 1}{(K_1^2+K_2^2)\rho^2}}. \qquad (1.17)$$

Next we consider the analogue of this situation for particles with spin $1/2$, described by the relativistic Dirac equation, when analysis is much more complicated.

2. The Dirac equation, separation of the variables

We start with the general covariant form of the Dirac equation [11]; in the coordinate system (1.1) we use the diagonal tetrad then the Dirac equation takes the form

$$\left[\gamma^0 \frac{\partial}{\partial t} + \gamma^1 e^z \frac{\partial}{\partial x} + \gamma^2 e^z \frac{\partial}{\partial y} + \right.$$

$$\left. + \gamma^3 \left(\frac{\partial}{\partial z} - 1\right) + im\right]\Psi = 0. \qquad (2.1)$$

Note that the addition of -1 about the operator ∂_z can be removed by substituting $\Psi = e^z\psi$. The following three operators $i\partial_t, i\partial_x, i\partial_y$ commute with the wave operator: solutions can be searched in the form

$$\Psi^{\varepsilon,k_1,k_2} = e^{-i\varepsilon t} e^{ik_1 x} e^{ik_2 y} \begin{vmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{vmatrix}. \qquad (2.2)$$

Using the Dirac matrices in spinor basis, from (2.1) we find equations for $f_i(z)$

$$-i\varepsilon f_3 - ik_1 e^z f_4 - k_2 e^z f_4 - (\frac{d}{dz} - 1) f_3 + imf_1 = 0,$$

$$-i\varepsilon f_4 - ik_1 e^z f_3 + k_2 e^z f_3 + (\frac{d}{dz} - 1) f_4 + imf_2 = 0,$$

$$-i\varepsilon f_1 + ik_1 e^z f_2 + k_2 e^z f_2 + (\frac{d}{dz} - 1) f_1 + imf_3 = 0,$$

$$-i\varepsilon f_2 + ik_1 e^z f_1 - k_2 e^z f_1 - (\frac{d}{dz} - 1) f_2 + imf_4 = 0.$$

$$(2.3)$$

There is a generalized helicity operator which commutes with the operator of the wave equation:

$$\Sigma = \frac{1}{2}\left(e^z \gamma^2\gamma^3 \frac{\partial}{\partial x} + e^z \gamma^3\gamma^1 \frac{\partial}{\partial y} + \right.$$

$$\left. + \gamma^1\gamma^2 (\frac{\partial}{\partial z} - 1)\right). \qquad (2.4)$$

Using the substitution (2.2) in the eigenvalues equation $\Sigma\Psi = p\Psi$ we obtain

$$k_1 e^z f_2 - ik_2 e^z f_2 - i(\frac{d}{dz} - 1) f_1 = pf_1,$$

$$k_1 e^z f_1 + ik_2 e^z f_1 + i(\frac{d}{dz} - 1) f_2 = pf_2,$$

$$k_1 e^z f_4 - ik_2 e^z f_4 - i(\frac{d}{dz} - 1) f_3 = pf_3,$$

$$k_1 e^z f_3 + ik_2 e^z f_3 + i(\frac{d}{dz} - 1) f_4 = pf_4. \qquad (2.5)$$

From equations (2.5) and (2.3), considered together, it follows a linear homogeneous system with respect to f_i

$$-i\varepsilon f_3 - ipf_3 + imf_1 = 0,$$
$$-i\varepsilon f_4 - ipf_4 + imf_2 = 0,$$
$$-i\varepsilon f_1 + ipf_1 + imf_3 = 0,$$
$$-i\varepsilon f_2 + ipf_2 + imf_4 = 0. \qquad (2.6)$$

We find two values for the p and the corresponding restrictions on the functions f_i:

$$p = \pm\sqrt{\varepsilon^2 - m^2},\ f_3 = \frac{\varepsilon - p}{m} f_1,\ f_4 = \frac{\varepsilon - p}{m} f_2. \qquad (2.7)$$

Thus, we have three continuous quantum number ε, k_1, k_2 and one discrete, which distinguishes the values $p = \pm\sqrt{\varepsilon^2 - m^2}$. In view of (2.7), from four equations (2.3) we arrive at two equations for f_1, f_2

$$(\frac{d}{dz} - 1 - ip) f_1 + e^z (ik_1 + k_2) f_2 = 0,$$
$$(\frac{d}{dz} - 1 + ip) f_2 - e^z (ik_1 - k_2) f_1 = 0. \qquad (2.8)$$

Note the symmetry of the equations with respect to change

$$f_1 \Rightarrow f_2, \qquad p \Rightarrow -p. \qquad (2.9)$$

It is convenient to obtain solutions of similar equations in the flat space

$$(\frac{d}{dz} - ip) f_1 + (ik_1 + k_2) f_2 = 0,$$
$$(\frac{d}{dz} + ip) f_2 - (ik_1 - k_2) f_1 = 0, \qquad (2.10)$$

so that

$$f_2 = -\frac{1}{ik_1 + k_2} (\frac{d}{dz} - ip) f_1,$$
$$\left(\frac{d^2}{dz^2} + \varepsilon^2 - m^2 - k_1^2 - k_2^2 \right) f_1 = 0. \qquad (2.11)$$

Here, there exist two independent solutions (let $k_3 = +\sqrt{\varepsilon^2 - m^2 - k_1^2 - k_2^2}$)

$$f_1^{(1)} = e^{+ik_3 z},\ f_2^{(1)} = -\frac{(+ik_3 - ip)}{ik_1 + k_2} e^{+ik_3 z},$$
$$f_1^{(2)} = e^{-ik_3 z},\ f_2^{(2)} = -\frac{(-ik_3 - ip)}{ik_1 + k_2} e^{-ik_3 z}. \qquad (2.12)$$

The sign before k_3 determines the direction of the wave propagation, the sign of p defines the state of polarization. Generalized analogue of these solution are to be investigated in the hyperbolic space H_3.

3. A special case of the waves along the z-axis

There exists a special case when $k_1 = 0,\ k_2 = 0$:

$$\Psi^{\varepsilon,0,0}(t,z) = e^{-i\varepsilon t} \begin{vmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{vmatrix}. \qquad (3.1)$$

The equations change substantially, and the problem reduces to

$$(\frac{\partial}{\partial z} - 1 - ip) f_1 = 0,\ f_1 = C_1 e^z e^{+ipz},$$
$$(\frac{\partial}{\partial z} - 1 + ip) f_2 = 0,\ f_2 = C_2 e^z e^{-ipz}. \qquad (3.2)$$

Solutions more simple to interpret are

$$\Psi^{\varepsilon 00 p}(t,z) = \begin{vmatrix} 1 \\ 0 \\ \frac{\varepsilon - p}{m} \\ 0 \end{vmatrix} e^z e^{+ipz}; \qquad (3.3a)$$

$$\Psi^{\varepsilon 00 p}(t,z) = \begin{vmatrix} 0 \\ 1 \\ 0 \\ \frac{\varepsilon - p}{m} \end{vmatrix} e^z e^{-ipz}. \qquad (3.3b)$$

Obviously, the factor e^z in the solutions will be compensated when considering any bilinear structure of the wave functions (with their subsequent multiplication by $\sqrt{-g}\ dx\, dy\, dz$).

4. Construction of solutions in the general case

Let us turn to (2.8) and introduce a new variable $\sqrt{k_1^2+k_2^2}e^z = Z, Z\in(0,+\infty),$:

$$\left(Z\frac{d}{dZ}-1-ip\right)f_1 + Z\sqrt{\frac{k_2+ik_1}{k_2-ik_1}}f_2 = 0, \quad (4.1a)$$

$$\left(Z\frac{d}{dZ}-1+ip\right)f_2 + Z\sqrt{\frac{k_2-ik_1}{k_2+ik_1}}f_1 = 0. \quad (4.1b)$$

From (4.1) we get two second order differential equations for f_1 and f_2:

$$Z\frac{d^2 f_1}{dZ^2} - 2\frac{d\, f_1}{dZ} + \left(\frac{p^2+ip+2}{Z} - Z\right)f_1 = 0, \quad (4.2)$$

$$Z\frac{d^2 f_2}{dZ^2} - 2\frac{d\, f_2}{dZ} + \left(\frac{p^2-ip+2}{Z} - Z\right)f_2 = 0. \quad (4.3)$$

Considering eq. (4.2), let us use a substitution $f_1(Z) = Z^A e^{BZ} F_1(Z)$; at A and B chosen according

$$A = +ip+1, -ip+2, B = \pm 1, \quad (4.4)$$

equation for F_1 becomes simpler (we use else one change $Z = y/2$ and let $B=-1$), then we arrive at equation for the confluent hypergeometric function (for definiteness let it be $A=+ip+1$)

$$y\frac{d^2\Phi}{dy^2} + (c-y)\frac{d\,\Phi}{dy} - a\Phi = 0,$$
$$a = +ip, c = 2a = +2ip. \quad (4.5)$$

Two linearly independent solutions are [12]

$$F_1^{(1)}(y) = \Phi(a,c,y),$$
$$F_1^{(2)}(y) = y^{1-c}\Phi(a-c+1, 2-c, y). \quad (4.6)$$

Consider the equation (4.3). Using the above-noted symmetry, we obtain

$$f_2 = y^{a'+1}e^{-y/2}F_2(y),\ a' = -ip, c' = -2ip,$$
$$F_2^{(1)} = \Phi(a',c',y),$$
$$F_2^{(2)} = y^{1-c'}\Phi(a'-c'+1, 2-c', y). \quad (4.7)$$

It is convenient to employ one independent parameter a:

$$f_1 = y^{a+1}e^{-y/2}F_1(y),$$
$$F_1^{(1)}(y) = \Phi(a,2a,y),$$
$$F_1^{(2)}(y) = y^{1-2a}\Phi(1-a, 2-2a, y); \quad (4.8a)$$

$$f_2 = y^{-a+1}e^{-y/2}F_2(y),$$
$$F_2^{(1)}(y) = \Phi(-a,-2a,y),$$
$$F_2^{(2)}(y) = y^{1+2a}\Phi(1+a, 2+2a, y). \quad (4.8b)$$

The functions f_1, f_2 (note that before now we did not find possible numerical factors at them) must be related by the first-order operators (4.1). These equations relate functions in the following pairs

$$F_1^{(1)}(y) - F_2^{(2)}(y), \qquad F_1^{(2)}(y) - F_2^{(1)}(y).$$

corresponding relative factors are calculated – the result is

$$I \qquad f_1 = M_+ e^{-y/2}y^{1+a}\,\Phi(a,2a,y),$$
$$f_2 = e^{-y/2}y^{2+a}\Phi(a+1, 2+2a, y),$$
$$M_+ = 2e^{+i\gamma}(1+2a); \quad (4.9)$$

$$II \qquad f_1 = M_- e^{-y/2}y^{2-a}\Phi(1-a, 2-2a, y),$$
$$f_2 = e^{-y/2}y^{1-a}\Phi(-a,-2a,y),$$
$$M_- = 2e^{-i\gamma}(1-2a), \quad (4.10)$$

where

$$e^{i\gamma} = \sqrt{\frac{k_2+ik_1}{k_2-ik_1}}.$$

Remind that $a = ip = \pm i\sqrt{\varepsilon^2 - m^2}$; the sign of p is associated with the polarization state of the spinor waves; types I and II are supposed to be associated with the directions of wave propagation: to the left or to the right.

To conclude this section we consider the limiting process in the constructed solutions to the case of the flat space. This will allow a better understanding of the obtained results in the Lobachevsky space. To this end, we first need to go to the usual dimensional quantities:

$$z = \frac{z_3}{R}, m = \frac{McR}{\hbar}, \varepsilon = \frac{ER}{c\hbar},$$

$$p = +R\sqrt{E^2/c^2\hbar^2 - M^2c^2/\hbar^2} = Rp_0,$$

$$k_1 = \frac{P_1 R}{c\hbar}, k_2 = \frac{P_2 R}{c\hbar}, \sqrt{k_1^2+k_2^2} = R\frac{\sqrt{P_1^2+P_2^2}}{c\hbar} = RK_\perp,$$

$$a = ip = iRp_0, \qquad c = 2a = i2Rp_0,$$

$$y = 2\sqrt{k_1^2+k_2^2}e^z = 2RK_\perp(1+\frac{x_3}{R}+\ldots) \longrightarrow 2RK_\perp.$$

Let us consider the solutions (4.9)

$$I \qquad f_1 = M_+ e^{-y/2}y^{1+a}\,\Phi(a,2a,y),$$

$$f_2 = e^{-y/2} y^{2+a} \Phi(a+1, 2+2a, y),$$
$$M_+ = \left[2 e^{+ia} (1+2a) \right];$$

taking into account

$$\frac{a}{c} y \Rightarrow RK_\perp,$$
$$\frac{1}{2!} \frac{a(a+1)}{c(c+1)} y^2 \Rightarrow \frac{1}{2!} (RK_\perp)^2,$$
$$\frac{1}{3!} \frac{a(a+1)(a+2)}{c(c+1)(c+2)} y^2 \Rightarrow \frac{1}{3!} (RK_\perp)^3 ...,$$

we get

$$e^{-y/2} \Rightarrow e^{-RK_\perp}, \Phi(a, 2a, y) \Rightarrow e^{RK_\perp},$$
$$e^{-y/2} \Rightarrow e^{-RK_\perp}, \Phi(a+1, 2a+2, y) \Rightarrow e^{RK_\perp},$$

and further

$$I \qquad f_1 \Rightarrow M_+ (2RK_\perp e^z)^{1+iRp_0} \sim e^{ix_3 p_0},$$
$$f_2 \Rightarrow (2RK_\perp e^z)^{2+iRp_0} \sim e^{ix_3 p_0}. \tag{4.11}$$

Similarly, we find

$$II \qquad f_2 = e^{-y/2} y^{1-a} \Phi(-a, -2a, y) \sim e^{-ix_3 p_0},$$
$$f_1 = M_- e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y) \sim e^{-ix_3 p_0}. \tag{4.12}$$

We may conclude that solutions of the type I (in curved model H_3) provide us with extension for the flat waves in Minkowski space of the type e^{+ikz}; whereas solutions of the type II represent extension for the flat waves in Minkowski space of the type e^{-ikz}.

Conclusions

In the paper complete systems of exact solutions for Schrödinger and Dirac equations in the hyperbolic space H_3 are constructed on the base of the method of separation of the variables in quasi-cartezian coordinates. An extended helicity operator is introduced. It is shown that solutions constructed when translating to the limit of vanishing curvature coincide with common plane wave solutions on Minkowski space going in opposite z-directions. Results are much the same for 2-component Weyl equation.

Author is grateful to V.M. Red'kov for encouragement and advices.

REFERENCES

1. Shapiro I.S. Expansion of the scattering amplitude in relativistic spherical functions. Phys. Lett. – 1962. – Vol. 1, No 7. – P. 253–255.
2. Vilenkin N.Ya., Smorodinsky Ya.A. Invariant Expansions of Relativistic Amplitudes. Soviet Physics JETP. – 1964. – Vol. 19. – P. 1209.
3. Gelfand I.M., Graev M.I., Vilenkin N.Y. Integral geometry and representation theory. Moscow, 1962.
4. Volobuev I.P. Plane waves on a sphere and some applications. Theoret. and Math. Phys. – 1980. – Vol. 45, No 3. – P. 1119–1122.
5. Ovsiyuk E.M., Tokarevskaya N.G., Red'kov V.M. Shapiro's plane waves in spaces of constant curvature and separation of variables in real and complex coordinates. NPCS. – 2009. – Vol. 12, No 1. – P. 1–15.
6. Bychkovskaya E.M. Solutions of Maxwell's equations in three-dimensional Lobachevsky space. Proceedings of the National Acad. of Sc. of Belarus. Series of physical-mathematical sciences. – 2006. – No 5. – P. 45–48.
7. Bogush A.A., Kurochkin Yu.A., Otchik V.S., Bychkovskaya E.M. Analogue of the plane electromagnetic waves in the Lobachevsky space. Non-euclidean geometry in modern physics: Proceedings of the Inter. Conf. BGL-5, Minsk, October 10-13, 2006 / National Academy of Sciences of Belarus, B.I. Stepanov Institute of Physics; Eds.: Yu. Kurochkin, V. Red'kov. – Minsk, 2006. – P. 111–115.
8. Bychkovskaya E.M. On wave solutions of Maxwell's equations in three-dimensional Lobachevsky space. Modern Problems in Physics. – Minsk, 2006. – P. 98–102.
9. Ovsiyuk E.M., Red'kov V.M. On solutions of the Maxwell equations in quasicartesian coordinates in Lobachebsky space. Proceedings of the National Acad. of Sc. of Belarus. Series of physical-mathematical sciences. – 2009. – No 4. – P. 99–105.
10. Kurochkin Yu.A., Otchik V.S. Solutions of the Dirac equation in the Lobachevsky

space. Proceedings of the National Academy of Sciences of Belarus. Series of physical-mathematical sciences. 2011. – No 2. – P. 31–35.
11. Red'kov V.M. The fields of the particles in a Riemannian space and the Lorentz group. Minsk, 2009.
12. Bateman G., Erdei A. Higher Transcendental Functions. Vol. 1. Hypergeometric function, Legendre functions. Moscow, 1973.
13. Kratzer A., Frantz V. Transcendental Functions. Moscow, 1963.
14. Ovsiyuk E.M., Red'kov V.M. On simulating a medium with special reflecting properties by Lobachevsky geometry (One exactly solvable electromagnetic problem). http://arxiv.org/abs/1109.0126.

Стаття надійшла до редакції 25.06.2012

Є. Овсіюк
Мозирський державний педагогічний університет, Мозир, Білорусь
e-mail: *e.ovsiyuk @ mail.ru*

УЗАГАЛЬНЕНІ ПЛОСКІ ХВИЛІ ДЛЯ ЧАСТИНОК ШРЕДІНГЕРА І ДІРАКА НА ФОНІ ГЕОМЕТРІЇ БОЯІ-ЛОБАЧЕВСЬКОГО

Показано, що геометрія Бояі-Лобачевського робить істотний вплив на квантово-механічну частинку, моделюючи середовище зі спеціальними відбиваючими властивостями ідеального дзеркала. Для скалярної частинки Шредінгера задача зведена до диференціального рівняння другого порядку, яке може бути зпівставлено з одномірним рівнянням Шредінгера для частинки в зовнішньому потенціальному полі $U(z) = U_0 e^{2z}$. У квантовій механіці викривлена геометрія виступає як ефективний потенціальний бар'єр з коефіцієнтом відбивання $R = 1$. Геометрія моделює середовище, ефективно діюче як ідеальне дзеркало. Проникнення частинок в ефективне середовище залежить від параметрів хвильових рішень $\varepsilon,\ k_1^2 + k_2^2$, а також від радіусу кривизни ρ. Аналогічний аналіз проведено для діраківської частинки зі спіном ½. В цьому випадку виникає додаткове квантове число, пов'язане з узагальненим оператором спіральності.

Ключові слова: геометрія Бояі-Лобачевського, плоска хвиля, рівняння Шредінгера, рівняння Дірака, гіпергеометричні функції.

Е. Овсиюк
Мозырский государственный педагогический университет, г. Мозыр, Беларусь
e-mail: *e.ovsiyuk@mail.ru*

ОБОБЩЕННЫЕ ПЛОСКИЕ ВОЛНЫ ДЛЯ ЧАСТИЦ ШРЕДИНГЕРА И ДИРАКА НА ФОНЕ ГЕОМЕТРИИ БОЙЯИ–ЛОБАЧЕВСКОГО

Показано, что геометрия Бояйи–Лобачевского оказывает существенное влияние на квантово-механическую частицу, моделируя среду со специальным отражающим свойствами идеального зеркала. Для скалярной частицы Шредингера задача сведена к дифференциальному уравнению второго порядка, которое может быть сопоставлено с одномерным уравнением Шредингера для частицы во внешнем потенциальном поле $U(z) = U_0 e^{2z}$. В квантовой механике искривленная геометрия выступает как эффективный потенциальный барьер с коэффициентом отражения $R = 1$. Геометрия моделирует среду, эффективно действующую как идеальное зеркало. Проникновение частиц в эффективную среду зависит от параметров волновых решений $\varepsilon,\ k_1^2 + k_2^2$, а также радиуса кривизны ρ. Аналогичный анализ проведен для дираковской частицы со спином ½, в этом случае возникает дополнительное квантовое число, связанное с обобщенным оператором спиральности.

Ключевые слова: геометрия Бояйи–Лобачевского, плоская волна, уравнение Шредингера, уравнение Дирака, гипергеометрические функции.